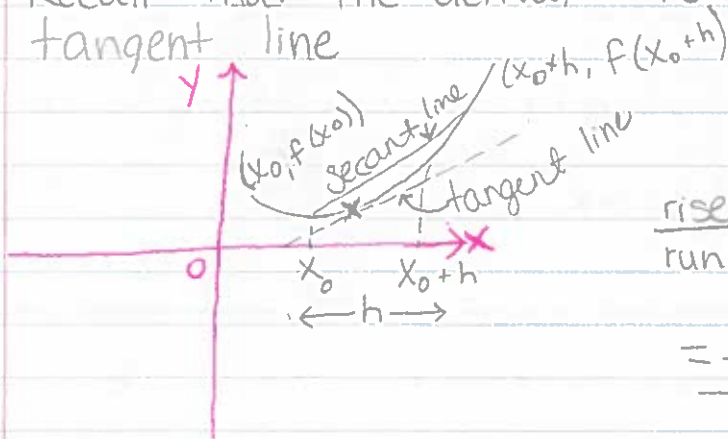


3.1 The Derivative as a limit

Recall that the derivative is the slope function of the tangent line



Slope of a secant line

$$\frac{\text{rise}}{\text{run}} = \frac{f(x_0+h) - f(x_0)}{x_0+h - x_0}$$

$$= \frac{f(x_0+h) - f(x_0)}{h}$$

As $(x_0+h, f(x_0+h))$ approaches $(x_0, f(x_0))$, we have

↳ secant line → tangent line at x_0

↳ slope of secant line

→ slope of tangent line at x_0

∴ slope of tangent line at x_0

$$\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h}$$

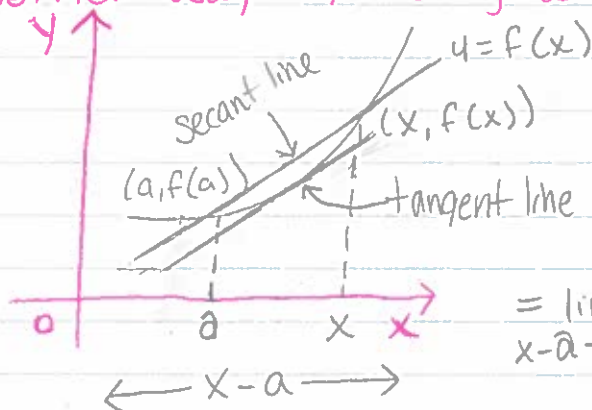
Notation: Slope of tangent line to $y=f(x)$ at x_0

i) $f'(x_0) \dots$

ii) $\left. \frac{dy}{dx} \right|_{x_0} \dots$

iii) $\frac{dy}{dx}(x_0) \dots$

Another way of looking at it...



as $(x, f(x))$ approaches $(a, f(a))$

↳ secant line → tangent line

↳ slope of secant line

→ slope of tangent at a

∴ slope of tangent line at a

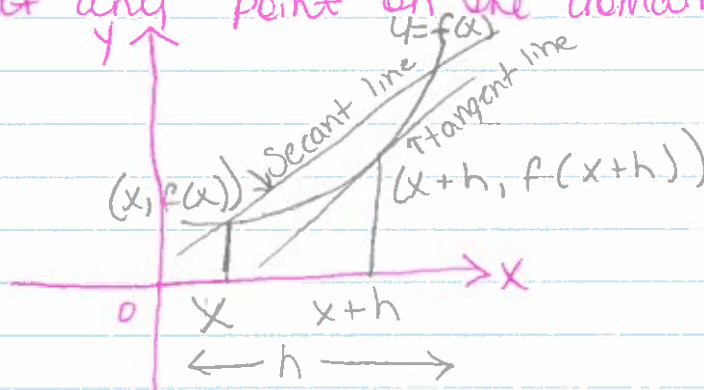
$$= \lim_{x-a \rightarrow 0} \frac{f(x) - f(a)}{x-a} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x-a}$$

Ch3 p2

ma121

may 24 18

Can we obtain a slope function $f'(x)$ or $\frac{dy}{dx}$ where it tells us the slope of the tangent at any point on the domain? How?



∴ The slope function is

$$f'(x) = \frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

↑ "First Principles" or "Definition of the derivative"

Example 10 p 71

Find the equation of the tangent line to $y = -\frac{2}{x}$ at $x=4$

To find the equation of a line, we need

↳ the slope, and,

↳ any point on the line (ie) (x_1, y_1)

The equation of the line is

$$y - y_1 = m(x - x_1)$$

The slope function of $y = f(x)$ is

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

OR we use

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

To straightaway find the slope of the tangent at $x=a$

so, $f'(4) = \lim_{x \rightarrow 4} \frac{f(x) - f(4)}{x - 4}$

$$\lim_{x \rightarrow 4} \frac{-\frac{2}{x} - (-\frac{2}{4})}{x - 4}$$

$$\lim_{x \rightarrow 4} \frac{-\frac{2}{x} + \frac{1}{2}}{x - 4}$$

$$\lim_{x \rightarrow 4} \frac{-\frac{4}{2x} + \frac{x}{2x}}{x - 4}$$

$$\lim_{x \rightarrow 4} \frac{-\frac{4+x}{2x}}{x - 4}$$

$$\lim_{x \rightarrow 4} \frac{-4+x}{2x(x-4)}$$

$$\lim_{x \rightarrow 4} \frac{-4+x}{2x(x-4)}$$

$$\lim_{x \rightarrow 4} \frac{x-4}{2x(x-4)}$$

$$\lim_{x \rightarrow 4} \frac{1}{2x}$$

$$= \frac{1}{4(2)}$$

$$= \frac{1}{8}$$

∴ $m = \frac{1}{8}$

Ch 3.2 p4
3.2

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There are general functions $y = f(x)$ that have already been run through the first principles

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

here we obtain the general derivative rule for this general function. For example, let us consider the power function $y = x^n$

$$\begin{aligned} &= \dots \\ &= nx^{n-1} \end{aligned}$$

This gives us the Power Rule

$$\begin{aligned} y &= x^n \\ \rightarrow \frac{dy}{dx} &= nx^{n-1} \end{aligned}$$

we also have the constant multiple Rule

$$\begin{aligned} y &= Cf(x) \leftarrow C \text{ is constant} \\ \rightarrow \frac{dy}{dx} &= cf'(x) \end{aligned}$$

and the Sum and Difference Rule

$$\begin{aligned} y &= f(x) \pm g(x) \\ \rightarrow \frac{dy}{dx} &= f'(x) \pm g'(x) \end{aligned}$$

→ Corollary

$$\begin{aligned} \text{i) } y &= C \leftarrow C \text{ is constant} \\ \rightarrow \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \end{aligned}$$

$$\lim_{h \rightarrow 0} \frac{C - C}{h} = \lim_{h \rightarrow 0} \frac{0}{h}$$

$$\lim_{h \rightarrow 0} (0) = 0$$

$$\begin{aligned} \text{If } y &= C \\ \rightarrow \frac{dy}{dx} &= 0 \end{aligned}$$

Ch 3 p 3



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We can use the point where the tangent line touches the curve (x_1, y_1) so when $x_1 = 4$, we have

$$y_1 = -2/x_1 = -2/4 = -1/2$$

$$\therefore (x_1, y_1) = (4, -1/2)$$

$$\Rightarrow \text{plug into } y - y_1 = m(x - x_1)$$

$$= y - (-1/2) = 1/8(x - 4)$$

$$= y + 1/2 = 1/8x - 1/2$$

$$= y = 1/8x - 1/2 - 1/2$$

$$= y = 1/8x - 1$$

Example 11 p 71

$$C = F + Vq \quad F = \$1200$$

$$\text{and } V = 0.05q - 1$$

$$C = 1200 + (0.05q - 1)(q)$$

$$C(q) = 1200 + 0.05q^2 - q$$

$$MC(q) = \lim_{h \rightarrow 0} \frac{C(q+h) - C(q)}{h}$$

$$\lim_{h \rightarrow 0} \frac{(1200 + 0.05(q+h)^2 - (q+h)) - (1200 + 0.05q^2 - q)}{h}$$

$$\lim_{h \rightarrow 0} \frac{\cancel{1200} + 0.05(q^2 + 2qh + h^2) - \cancel{q} - h - \cancel{1200} - 0.05q^2 + \cancel{q}}{h}$$

$$\lim_{h \rightarrow 0} \frac{0.05(q^2 + 2qh + h^2) - h - 0.05q^2}{h}$$

$$\lim_{h \rightarrow 0} \frac{\cancel{0.05q^2} + 0.1qh + 0.05h^2 - h - \cancel{0.05q^2}}{h}$$

$$\lim_{h \rightarrow 0} \frac{0.1qh + 0.05h^2 - h}{h}$$

$$\lim_{h \rightarrow 0} \frac{h(0.1q + 0.05h - 1)}{h}$$

$$\lim_{h \rightarrow 0} (0.1q + 0.05h - 1)$$

$$= 0.1q + 0.05(0) - 1$$

$$MC(q) = 0.1q - 1$$

$$\text{when } MC(q) = 0,$$

$$0.1q - 1 = 0$$

$$\rightarrow 0.1q = 1$$

$$\rightarrow q = \frac{1}{0.1}$$

$$\rightarrow q = 10$$

Hilary

Ch 3 PS MA 121

May 24/18

ii) $y = cx$ $\leftarrow C$ is constant

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{c(x+h) - cx}{h}$$

$$\lim_{h \rightarrow 0} \frac{cx + ch - cx}{h}$$

$$\lim_{h \rightarrow 0} \frac{ch}{h}$$

$$\lim_{h \rightarrow 0} c = c$$

If $y = cx$

$$\rightarrow \frac{dy}{dx} = c$$

Example 3 p 75

$$y = 4.5x^5 + 2x^{-3} + x^2 + 7.2x + 11.1$$

$$\rightarrow \frac{dy}{dx} = -4.5(5x^{5-1}) + 2(-3x^{-3-1}) + 2x^{2-1} + 7.2 + 0$$

$$= -22.5x^4 - 6x^{-4} + 2x + 7.2$$

remember
 $y = x^n$
 $\rightarrow \frac{dy}{dx} (nx^{n-1})$

the
 derivative
 of a
 constant is
 ALWAYS

0

4 p 75

$$y = 4/9 x^9 - 2x^{3/2} + 14/3$$

$$\rightarrow \frac{dy}{dx} = 4/9(9x^{9-1}) - 2(3/2 x^{3/2-1}) + 0$$

$$= 4x^8 - 3x^{1/2}$$

Notation

$$i) \sqrt[n]{x^m} = (\sqrt[n]{x})^m = x^{\frac{m}{n}}$$

$$ii) \frac{1}{x^n} = x^{-n}$$

$$iii) x^m x^n = x^{m+n}$$

$$iv) \frac{x^m}{x^n} = x^{m-n}$$

Ch 3 P6

MA121

Example

17 P75

$$\begin{aligned}
 y &= \sqrt{x} (5 - x^3) \\
 &= 5\sqrt{x} - x^3(\sqrt{x}) \\
 &= 5x^{1/2} - x^3 x^{1/2} \\
 &= 5x^{1/2} - x^{3+1/2} \\
 y &= 5x^{1/2} - x^{7/2} \\
 \frac{dy}{dx} &= 5\left(\frac{1}{2}x^{1/2-1}\right) - \frac{7}{2}x^{7/2-1} \\
 &= \frac{5}{2}x^{-1/2} - \frac{7}{2}x^{5/2}
 \end{aligned}$$

20 P75

$$\begin{aligned}
 y &= \frac{2x(x + 10x^8)}{x^2} \\
 &= \frac{2(x + 10x^8)}{x} \\
 &= 2\left(\frac{x}{x} + \frac{10x^8}{x}\right) \\
 &= 2(1 + 10x^7) \\
 \frac{dy}{dx} &= 2(0 + 10(7x^{7-1})) \\
 &= 2(70x^6) \\
 &= 140x^6
 \end{aligned}$$

36 P76

$$C(q) = 0.0005q^3 - 0.01q^2 + 40q + 990 \quad [0, 500]$$

$$\begin{aligned}
 \text{i) } MC(q) &= 0.0005(3q^{3-1}) - 0.01(2q^{2-1}) + 40 + 0 \\
 &= 0.0015q^2 - 0.02q + 40
 \end{aligned}$$

$$MC(300) = 0.0015(300)^2 - 0.02(300) + 40$$

$$\text{when } q = 300, MC = \$169$$

$$\text{ii) avg. cost } A(q) = \frac{C(q)}{q}$$

$$= \frac{0.0005q^3 - 0.01q^2 + 40q + 990}{q} = 0.0005q^2 - 0.01q + 40 + \frac{990}{q}$$

may 24 18

36 continued

$$0.0005q^2 - 0.01q + 40 + 990q^{-1}$$

$$\begin{aligned}
 \text{iii) } MA(q) &= 0.0001q - 0.01 - 990q^{-2} \\
 &= 0.001q - 0.01 - \frac{990}{q^2}
 \end{aligned}$$

$$\text{At } q = 100,$$

$$\begin{aligned}
 MA(q) &= 0.001(100) - 0.01(100) \\
 &\quad - \frac{990}{100^2}
 \end{aligned}$$

$$= 0.009$$

$$= 0.9 \text{ cents.}$$

continued